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DOMINATION AND COLORING IN GRAPHS

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Abstract: Graph coloring theory and domination in graphs are two major areas within graph theory which have been extensively studied. A set $S \subset V$ is said to be a dominating set of G if every vertex $v \in V - S$ is adjacent to a vertex in S. If further, S is independent, then S is called an independent dominating set of G. The minimum cardinality of an independent dominating set is called independent domination number of G and is denoted by $i(G)$.The fundamental parameter in the theory of graph coloring is the chromatic number $\chi(G)$ of a graph G which is defined to be the minimum number of colors required to color the vertices of G in such a way that no two adjacent vertices of G receive the same color. A vertex $v \in V$ is a dominator of a set $S \subseteq V$ if v dominates every vertex in S. A partition $\Pi = \{V_1, V_2, ..., V_k\}$ is called a dominator partition if every vertex $v \in V$ is a dominator of at least one V_i. The dominator partition number $\Pi_d(G)$ equals the minimum k such that G has a dominator partition of order k.
If we further require that Π be a proper coloring of G, then we have a dominator coloring of G. The dominator chromatic number $\chi_d(G)$ is the minimum number of colors required for a dominator coloring of G. We present some variations of this parameter and several interesting results and unsolved problems on them.

Keywords and Phrases: Fall-chromatic number, b-chromatic number, dominator chromatic number, global dominator chromatic number, min-dom-color number.

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1. Introduction

By a graph $G = (V, E)$, we mean a connected, finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Harary [5]. Graph coloring theory and domination in graphs are two major areas within graph theory which have been extensively studied. Graph coloring deals with the fundamental problem of partitioning a set of objects into classes, according to certain rules. Time tabling, sequencing and scheduling problems in their many terms are basically of this nature. The fundamental parameter in the theory of graph coloring is the chromatic number $\chi(G)$ of a graph G which is defined to be the minimum number of colors required to color the vertices of G in such a way that no two adjacent vertices of G receive the same color. If $\chi(G) = k$, we say that G is k-chromatic. The chromatic number is a very well studied parameter whose history dates back to the famous four color problem and the early work of Kempe [9] in 1879 and Heawood [7] in 1890. Another fastest growing area within graph theory is the study of domination and related subset problems such as independence, covering and matching. A comprehensive treatment of the fundamentals of domination is given in Haynes et al. [6]. A set $S \subset V$ is said to be a dominating set of G if every vertex $v \in V - S$ is adjacent to a vertex in S. If further, S is independent, then S is called an independent dominating set of G. The minimum cardinality of an independent dominating set is called the independent domination number of G and is denoted by $i(G)$. An independent dominating set of G of cardinality $i(G)$ is called an $i(G)-$ set of G. A partition of $V(G)$ into independent dominating sets of G is called an independent domatic partition of G .
A vertex $v \in V$ is a dominator of a set $S \subseteq V$ if v dominates every vertex in S. A partition $\Pi = \{V_1, V_2, ..., V_k\}$ is called a dominator partition if every vertex $v \in V$ is a dominator of at least one V_i. The dominator partition number $\Pi_d(G)$ equals the minimum k such that G has a dominator partition of order k.
If we further require that Π be a proper coloring of G, then we have a dominator coloring of G. The dominator chromatic number $\chi_d(G)$ is the minimum number of colors required for a dominator coloring of G. In this paper we initiate a study of this parameter and also some variations of the same.

2. Main Results

There are many parameters relating domination and colorings.We shall define some of them. Dunbar et al.defined fall chromatic number of a graph.

Definition 2.1. *A coloring $C = \{V_1, V_2, \ldots, V_k\}$ is called a fall coloring if $N[v] \cap V_i \neq \phi$ for all $v \in V$ and for every i. If a graph has a fall coloring then every color class is a dominating set. Minimum k is fall chromatic number $\chi_f(G)$.*

Definition 2.2. *A b-coloring of a graph G is a Coloring of the vertices in such a way that each color class contains a vertex that has a neighbor in all other color classes. Maximum i is b-chromatic number $\chi_b(G)$.*

Walikar et al. [9] have proved the following theorem.

Theorem 2.3. *In a k-chromatic graph G, any k-coloring of G yields another k-coloring of G containing a color class which is a dominating set of G.*

Hence it is natural to consider the maximum number of color classes which are dominating sets of G, where the maximum is taken over all k-colorings of G, which we call the dom-color number of G.

Definition 2.4. *d_c = Number of color classes that are dominating sets. Dom chromatic number $d_\chi(G) = max_C d_C$.*

Chromatic transversal domination is defined as follows:

Definition 2.5. *A dcc -set is a dominating set which intersects every color class of every $\chi-$ coloring of G.Minimum cardinality is the chromatic transversal domination number.*

Motivated by a paper of Cockayne and Hedetniemi, Gera et al. ([3, 4]) introduced dominator coloring in graphs.
Later Hedetniemi, Arumugam, Sahul Hameed and many others have made reasonable contributions to this.
Definition 2.6. *A vertex $v \in V$ is a dominator of a set $S \subseteq V$ if v dominates every vertex in S. A partition $\Pi = \{V_1, V_2, ..., V_k\}$ is called a dominator partition if every vertex $v \in V$ is a dominator of at least one V_i. The dominator partition number $\Pi_d(G)$ equals the minimum k such that G has a dominator partition of order k.*

If we further require that Π be a proper coloring of G, then we have a dominator coloring of G. The dominator chromatic number $\chi_d(G)$ is the minimum number of colors required for a dominator coloring of G.
Since every vertex is a dominator of itself, the partition $\{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$ of $V(G)$ into singleton sets is a dominator coloring. Thus, every graph of order n has a dominator coloring of order n and so the parameter dominator chromatic number $\chi_d(G)$ is well defined.

Example 2.7. For the graph G in Fig 2.1, $\chi_d(G) = 3$.

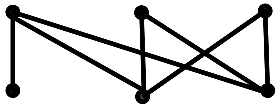

Fig 2.1

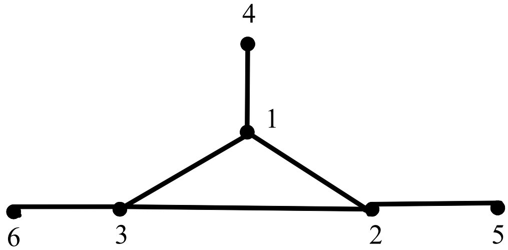

Fig 2.2

For the graph in Fig 2.2, $\chi = 3$ and $\chi_d = 4$.

Lemma 2.8. *1. Consider the Bistar $B_{m,n}$. For $m + n \geq 4, \chi_d(B_{m,n}) = 3$.*
2. $\chi_d(K_{1,n}) = 2$.
3. $\chi_d(K_n) = n$.
4. For $n \geq 2$,
$$\chi_d(P_n) = \begin{cases} 1 + \lceil \frac{n}{3} \rceil \; if n = 2, 3, 4, 5, 7 \\ 2 + \lceil \frac{n}{3} \rceil \; otherwise \end{cases}$$
5. For the wheel W_n,
$$\chi_d(W_n) = \begin{cases} 3 if \;\; n \;\; is \;\; even \\ 4 if \; n \; is \; odd \end{cases}$$
6. For the complete $k-$ partite graph $\chi_d(K_{a_1,a_2,...a_k}) = k$.
7. Star can be generalised to the multistar $K_m(a_1, a_2, ..., a_m)$ which is formed by joining $a_i \geq 1 (1 \leq i \leq m)$ vertices to each vertex x_i of a complete graph K_m with $V(K_m) = \{x - 1, x_2, ..., x_m\}$. We have $\chi_(K_n(a_1, a_2, ..., a_m)) = n + 1$.
8. For cycle C_n,
$$\chi_d(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil & if \;\; n = 4 \\ \lceil \frac{n}{3} \rceil + 1 & if \;\; n = 5 \\ \lceil \frac{n}{3} \rceil + 2 & otherwise \end{cases}$$
9. It has been conjectured that for Q_n, $\chi_d(Q_n) = 2 + 2^{n-2}$. The conjecture is false. For example, $\gamma(Q_7) = 16$. But $\chi_d(Q_7) \leq \gamma(Q_7) + \chi(Q_7) = 16 + 2 = 18$.

Remark 2.9. *1. It is always true that $\chi(G) \leq \chi_d(G)$ for any graph G. Both strict inequality and equality are possible.*

For $B_{m,n}$, $\chi = 2$ and $\chi_d = 3$. For $K_{1,n}$, $\chi = \chi_d = 2$.
2. For a given graph G and a subgraph H, the dominator chromatic number of subgraph H may be smaller or larger than the dominator chromatic number of G.
If $G = K_n (n \geq 3)$ and $H = K_{a,b}, a + b = n$, $\chi_d(H) = 2$ whereas $\chi_d(G) = n$.
If $G = K_{n,n}$ and $H = P_{2n}$, $\chi_d(H) = \lceil \frac{n}{3} \rceil + 2 \geq 5$ whereas $\chi_d(G) = 2$.
3. For any connected graph G, $\chi_d(G) \leq n+1-\alpha(G)$ where $\alpha(G)$ is the independence number of G. For sharpness of the bound, consider $B_{m,n}$; $\alpha(B_{m,n}) = m+n = p-2$ and $\chi_d = 3$.

Following are the theorems that are proved in [1].

Theorem 2.10. *For any connected graph G of order $n \geq 2$, $2 \leq \chi_d(G) \leq n$. Also (1)$\chi_d(G) = 2$ iff $G \equiv K_{a,b}$ for $a, b \in \mathbb{Z}$.*
(2)$\chi_d(G) = n$ iff $G \equiv K_n$ for $n \in \mathbb{Z}$.

Theorem 2.11. *Let G be a graph of order n. Then $\chi_d(G) = n$ iff $G = K_a \cup (n - a)K_1$ where $1 \leq a \leq n$.*

Theorem 2.12. *Let G be a connected graph. Then $max\{\chi(G), \gamma(G)\} \leq \chi_d(G) \leq \chi(G) + \gamma(G)$. Furthermore the bounds are sharp. Lower bound is attained for Complete bipartite graphs and upper bound is attained for even cycles $C_n, n \geq 8$.*

Theorem 2.13. *Let G be a connected graph. If $\gamma(G) = 1$, then $\chi(G) = \chi_d(G)$ and every pair $(1, a)$ with $a \geq 1$ is realizable as $\gamma(G) = 1$ and $\chi(G) = \chi_d(G) = a$.*

Theorem 2.14. *For each ordered triple of integers $(a, b, c)(c > a > 1, c > b \geq 2, c \leq a+b)$, there is a connected graph G with $\gamma(G) = a, \chi(G) = b$ and $\chi_d(G) = c$, except $(1, b, b + 1)$.*

Theorem 2.15. *Let G be any graph with $\delta(G) = 1$. Then $\chi_d(G) > \gamma(G)$.*

In search for triangle-free graphs with arbitrary large chromatic number, Mycielski gave an elegant graph transformation.

Theorem 2.16. *For a graph $G = (V, E)$, the Mycielskian of G is the graph $\mu(G)$ with vertex set $V \cup V' \cup \{u\}$ where $V' = \{x' : x \in V\}$ and is disjoint from V. $E' = E \cup \{xy' : xy \in E\} \cup \{x'u : x' \in V'\}$. The vertices x and x' are called twins of each other and u is called the root of $\mu(G)$.*

Let G be any graph and let $\mu(G)$ be the Mycielskian graph of G. Then $\chi_d(G)+1 \leq \chi_d(\mu(G)) \leq \chi_d(G) + 2$.

3. Global Dominator Chromatic Number of a Graph

I. Sahul Hamid and Rajeswari introduced Global dominator coloring of a graph.

Definition 3.1. *Let G be a graph and let $S \subseteq V$. A vertex $v \in V$ is a dominator of*

S if v dominates every vertex in S. A vertex $v \in V$ is said to be an anti-dominator of S if v dominates none of the vertices of S. In a coloring $\mathbb{C} = \{V_1, V_2, ..., V_k\}$ of a graph G, a color class V_i is called a dom-color class or an anti-dom-color class of a vertex v of G according as v is a dominator of V_i or an anti-dominator of V_i . With these terminologies, a dominator coloring can also be defined to be a coloring with the property that every vertex has a dom color class. We define a coloring $\mathbb{C}$ of G to be a global dominator coloring of a graph G if $\mathbb{C}$ has the property that every vertex of G has both a dom-color class and an anti-dom-color class in $\mathbb{C}$. The minimum number of colors required for a global dominator coloring of G is called the global dominator chromatic number of G and is denoted by χ_{gd}.

Example 3.2. The graph given in Fig 3.1 is one with $\chi(G) = 2, \chi_d(G) = 2$ and $\chi_{gd}(G) = 4$.

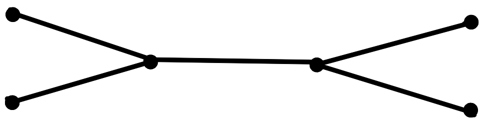

Fig 3.1

Example 3.3. 1. Always $\chi_d \leq \chi_{gd}$. If G is a connected graph with $\chi_d(G) \geq \Delta(G) + 2$, then $\chi_{gd}(G) = \chi_d(G)$.
2. For Petersen graph PG, $\chi_{gd}(PG) = 5$.
3. $\chi_{gd}(G) = 2$ iff $G \equiv \overline{K_2}$.

4. Dom-color Number of a Graph

Arumugam et al.defined dom-color number of a graph.

Definition 4.1. *Let G be a graph with $\chi(G) = k$. Let $\mathbb{C} = (V_1, V_2, ..., V_k)$ be a $k-$ coloring of G. Let $d_{\mathbb{C}}$ denote the number of color classes in $\mathbb{C}$ which are dominating sets of G. Then $d_\chi = max_{\mathbb{C}} d_{\mathbb{C}}$ where the maximum is taken over all $k-$ colorings of G, is called the dom-color number of G.*

Example 4.2. (i) $d_\chi(C_n) = \chi(C_n) = 2$ if n is even. If n is odd, then $\chi(C_n) = 3$ and $d_\chi(C_n) = 2$.
(ii) $d_\chi(K_n) = \chi(K_n) = n$.
(iii) For any connected bipartite graph G ,$d_\chi(G) = \chi(G) = 2$.

Theorem 4.3. $1 \leq d_\chi(G) \leq \chi(G)$ *for any graph G.*

Theorem 4.4. *Given integers a and b with $1 \leq a \leq b$, there exists a graph G with $d_\chi(G) = a$ and $\chi(G) = b$.*

If $a = 1$ then $G = K_b o K_1$.
If $a \geq 2$, G is obtained by concatenating a vertex of K_b with a vertex of K_a.

Theorem 4.5. *For any uniquely colorable graph G, $d_\chi(G) = \chi(G)$. Converse is not true.*

John Arul Singh and Kala defined the following:

Definition 4.6. *Let G be a graph with $\chi(G) = k$. Let $\mathcal{C} = \{V_1, V_2, \ldots, V_k\}$ be a k-coloring of G. Let $d_{\mathcal{C}}$ denote the number of color classes in $\mathcal{C}$ which are dominating sets of G. Then $md_\chi(G) = \overset{min}{\mathcal{C}} d_{\mathcal{C}}$, where the minimum is taken over all k-colorings of G, is called the min-dom-color number of G.*

Remark 4.7. *Min-dom-color number of G exists for all graphs and $0 \leq md_\chi(G) \leq \chi(G)$.*

Example 4.8. 1. $md_\chi(K_n)$=n.

$$2.\ \ md\chi(C_n) = \begin{cases} 0 & \text{if } n \geq 9 \text{ and n is odd} \\ 1 & \text{if } n = 7 \\ 2 & \text{if } n \text{ is even or } n = 5 \\ 3 & \text{if } n = 3 \end{cases}$$

3. $md_\chi(W_n) = md_\chi(C_n) + 1$.
4. For any complete k-partite graph G, $md_\chi(G) = k$.
5. For any uniquely colorable graph G, $md_\chi(G) = \chi(G)$. In particular, for a tree T, $md_\chi(T) = 2$.
6. If G is a bipartite graph without isolated vertices then $md_\chi(G) = 2$.
7. If G is a graph with at least two isolated vertices, then $md_\chi(G) = 0$.
8. If $\Delta(G) = n - 1$, then $md_\chi(G) \geq 1$.

Open Problems

1 Characterize graphs with $\chi_d(G) = \chi(G)$.

2 Characterize graphs with $\chi_d(G) = \gamma(G)$.

3 Characterize graphs with $\chi_d(G) = \gamma(G) + \chi(G)$.

4 Dominator chromatic number may increase arbitrarily on the removal of a vertex. For example, $\chi_d(W_n) = \chi_d(C_{n-1} + K_1) = 3$ or 4 according as n is odd or even and on removing the central vertex of W_n, χ_d increases arbitrarily. Hence the study of changing and unchanging of the dominator chromatic number on the removal of a vertex or an edge is an interesting problem.

5 Characterise the class of graphs with $md_\chi(G) = 0$.

6 Characterise the class of graphs with $md_\chi(G) = \chi(G)$.

7 Characterise the class of graphs with $md_\chi(G) = d_\chi(G)$.

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